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On the Linear Stability of Hydromagnetic Flow for Nonaxisymmetric Disturbances

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The linear stability of hydromagnetic inviscid flow of a heterogeneous fluid between two concentric cylinders is studied for nonaxisymmetric disturbances. A sufficient condition of stability has been derived for rigidly rotating fluid permeated by an azimuthal magnetic field against all disturbances. In the case of swirling flow permeated by an axial magnetic field, a bound on the growth rate is found without dependence of the wavenumbers. © 1987 Academic Press, Inc.

I. INTRODUCTION

In the study of linear stability of hydrodynamic flow, it has been observed (Pedley [1]; Lessen, Sedler, and Lin [2]), that nonaxisymmetric disturbances are the most unstable ones. Consequently, there have been several investigations to extend the results obtained by Howard & Gupta [3] and Howard [4] to nonaxisymmetric disturbances. It was possible to obtain a circle theorem for nonaxisymmetric disturbances in some flow situations (see, e.g., Barston [5]). In the general case, Barston's approach, however, yields a circle-theorem with the radius of the circle depending on the wavenumbers. Following Howard [4], Lalas [6] succeeded in establishing a suitable Richardson criterion to guarantee stability of compressible swirling flow against arbitrary disturbances. Using the method of

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Howard and Gupta [3], Kurzweg [7] derived two conditions to be satisfied simultaneously for stability of heterogeneous swirling flow. For homogeneous swirling flow, it was, however, not possible to define a Richardson number or to derive such a sufficient condition for stability against arbitrary disturbances.

In the hydromagnetic case, it is well known that the hydromagnetic effects are pronounced for nonaxisymmetric disturbances due to twisting of the lines of force. This is particularly true when the field is azimuthal. A natural extension to include the effect of magnetic field on flow stability for arbitrary disturbances has been attempted with partial success (Acheson [8, 9], Barston [5], Lucas [10], and Ganguly and Gupta [11]). No attempt has, however, been made to derive a suitable Richardson criterion for hydromagnetic flow against arbitrary disturbances. It is thus important to see whether the analysis of Howard [4] and Lalas [6] may be extended for hydromagnetic flows. The present work is addressed to that direction and employs a different method.

II. Swirling Flow in Presence of an Azimuthal Magnetic Field

We consider inviscid heterogeneous flow between two infinitely long concentric cylinders $r=a$ and $r=b$. The steady-state velocity, pressure and density of the flow are given by $[0, r\Omega(r), W(r)]$, $P_0(r)$ and $\rho_0(r)$, respectively, in cylindrical coordinates (r, θ, z). The flow is permeated by an azimuthal magnetic field $B_\theta(r)$, produced by a suitable distribution of current. We look for stability condition of this basic flow subjected to infinitesimal disturbances of the form

$$[\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_z, \bar{b}_r, \bar{b}_\theta, \bar{b}_z, \tilde{p}, \tilde{\rho}]$$
$$= [u_r(r), u_\theta(r), u_z(r), b_r(r), b_\theta(r), b_z(r), p(r), \rho(r)] \exp[i(\omega t - m\theta - kz)],$$

where the quantities with bar are actually functions of r, θ, z, and t, and ω is the complex frequency with m and k denoting the usual azimuthal and axial wavenumbers. We also introduce Lagrangian displacement vector $\xi(\xi_r, \xi_\theta, \xi_z)$ defined by

$$u_r = i\sigma\xi_r,$$

$$u_\theta = i\sigma\xi_\theta - r\frac{d\Omega}{dr}\xi_r,$$

$$u_z = i\sigma\xi_z - \frac{dW}{dr}\xi_r,$$

where

$$\sigma(r) = \omega - m\Omega(r) - kW(r).$$

The linearized perturbation equations are now in terms of ξ, (see Chandrasekhar [12])

$$\left\{\rho_0\sigma^2 - \rho_0 2\Omega r\frac{d\Omega}{dr} - \Omega^2 r\frac{d\rho_0}{dr} + \frac{1}{\mu}\left[2\frac{B}{r}\left(\frac{dB}{dr} - \frac{B}{r}\right) - \frac{m^2B^2}{r^2}\right]\right\}\xi_r$$

$$+\left\{2i\rho_0\Omega\sigma + \frac{2\,\mathrm{im}\,B^2}{\mu r^2}\right\}\xi_\theta - \frac{dp}{dr} = 0, \tag{2.1}$$

$$-\left\{2i\rho_0\Omega\sigma + \frac{2\,\mathrm{im}\,B^2}{\mu r^2}\right\}\xi_r + \left\{\rho_0\sigma^2 - \frac{m^2B^2}{\mu r^2}\right\}\xi_\theta + \frac{\mathrm{im}\,p}{r} = 0, \tag{2.2}$$

$$\left\{\rho_0\sigma^2 - \frac{m^2B^2}{\mu r^2}\right\}\xi_z + ikp = 0. \tag{2.3}$$

The equation of continuity gives

$$\frac{d\xi_r}{dr} + \frac{\xi_r}{r} - \frac{\mathrm{im}\,\xi_\theta}{r} - ik\xi_z = 0. \tag{2.4}$$

The boundary conditions corresponding to this flow are

$$\xi_r(a) = \xi_r(b) = 0. \tag{2.5}$$

Following a procedure similar to Lalas [6], we assume that the flow is unstable (i.e., ω_i, the imaginary part of ω, is nonzero) and introduce the transformation defined by

$$\xi_r = \sigma^{1/2}G_r, \tag{2.6}$$

$$\xi_\theta = \sigma^{1/2}G_\theta - \frac{\sigma^{1/2}}{|\sigma|^2}\omega_i r\frac{d\Omega}{dr}G_r, \tag{2.7}$$

$$\xi_z = \sigma^{-1/2}G_z - \frac{\sigma^{-1/2}}{|\sigma|^2}\omega_i\frac{dW}{dr}G_r. \tag{2.8}$$

With this set of transformations and by use of (2.4), one can easily verity that

$$\frac{1}{r}\frac{d}{dr}(\bar{\sigma}^{-1/2}rG_r) - \bar{\sigma}^{-1/2}\frac{\mathrm{im}\,G_\theta}{r} - \bar{\sigma}^{-1/2}ikG_z = 0. \tag{2.9}$$

We also have the form (2.5),

$$G_r(a) = G_r(b) = 0. \tag{2.10}$$

Substituting equations (2.6), (2.7), and (2.8) in Eqs. (2.1), (2.2), and (2.3), we get

$$(\sigma^2 L + \sigma M + N)\,\sigma^{-1/2} G + F = 0, \tag{2.11}$$

where

$$L = \begin{bmatrix} \rho_0 & 0 & 0 \\ -\rho_0 \dfrac{\omega_i}{|\sigma|^2} r \dfrac{d\Omega}{dr} & \rho_0 & 0 \\ -\rho_0 \dfrac{\omega_i}{|\sigma|^2} \dfrac{dW}{dr} & 0 & \rho_0 \end{bmatrix} \tag{2.12}$$

$$M = \begin{bmatrix} -2i\rho_0 \Omega \dfrac{\omega_i}{|\sigma|^2} r \dfrac{d\Omega}{dr} & 2i\rho_0 \Omega & 0 \\ -2i\rho_0 \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.13}$$

$$N = \begin{bmatrix} \begin{matrix} -2\rho_0 r\Omega \dfrac{d\Omega}{dr} - \Omega^2 r \dfrac{d\rho_0}{dr} \\ -\dfrac{m^2 B^2}{\mu r} + 2\dfrac{B}{\mu r}\left(\dfrac{dB}{dr} - \dfrac{B}{r}\right) \\ -2im \dfrac{B^2}{\mu r^2} \dfrac{\omega_i}{|\sigma|^2} r \dfrac{d\Omega}{dr} \end{matrix} & 2i \dfrac{mB^2}{\mu r^2} & 0 \\ -\dfrac{2\,\mathrm{im}\,B^2}{\mu r^2} + \dfrac{m^2 B^2}{\mu r^2} \dfrac{\omega_i}{|\sigma|^2} r \dfrac{d\Omega}{dr} & -\dfrac{m^2 B^2}{\mu r^2} & 0 \\ \dfrac{m^2 B^2}{\mu r^2} \dfrac{\omega_i}{|\sigma|^2} \dfrac{dW}{dr} & 0 & -\dfrac{m^2 B^2}{\mu r^2} \end{bmatrix} \tag{2.14}$$

and

$$F = \left[-\frac{dp}{dr}, \frac{\mathrm{im}}{r} p, ikp \right]^{\mathrm{T}} \tag{2.15}$$

with the superscript T denoting transpose.

We now introduce a function space E consisting of the set of all complex-valued three vectors $G = [G_r, G_\theta, G_z]$ such that each of the components appearing in (2.11) is a continuous function of r on the closed interval $[a, b]$. The function space is equipped with the inner product

$$(\xi, \eta) = \int_a^b (\bar{\xi}_r \eta_r + \bar{\xi}_\theta \eta_\theta + \bar{\xi}_z \eta_z)\, r\, dr,$$

for every $\xi, \eta \in E$ where $\bar{\xi}$ is the complex conjugate of ξ. Let Δ be the subspace of E such that $G_r(r)$ is continuously differentiable on $[a, b]$ and Eqs. (2.9) and (2.10) hold. Thus for any $G \in \Delta$, it follows from Eqs. (2.9) and (2.10),

$$\begin{aligned}(\bar{\sigma}^{-1/2} G, F) &= \int_a^b \left[-\sigma^{-1/2} \bar{G}_r \frac{dp}{dr} + \sigma^{-1/2} \bar{G}_\theta \frac{\mathrm{im}}{r} p + \sigma^{-1/2} \bar{G}_z ikp \right] r\, dr \\ &= -r\sigma^{-1/2} \bar{G}_r p|_a^b + \int_a^b \left[\frac{1}{r} \frac{d}{dr} (\sigma^{-1/2} r \bar{G}_r) \right. \\ &\quad \left. + \sigma^{-1/2} \frac{\mathrm{im}}{r} \bar{G}_\theta + \sigma^{-1/2} ik \bar{G}_z \right] pr\, dr \\ &= 0.\end{aligned}$$

The inner product of (2.11) with $\bar{\sigma}^{-1/2} G$ now yields

$$(\bar{\sigma}^{-1/2} G, [\sigma^2 L + \alpha M + N]\, \sigma^{-1/2} G) = 0$$

which simplifies to

$$\left(G, \left[\sigma L + M + \frac{N}{\sigma} \right] G \right) = 0. \tag{2.16}$$

It becomes calculationally convenient if we now split the operators appearing in Eqs. (2.12), (2.13), and (2.14) as

$$L = L_1 + iL_2, \; M = M_1 + iM_2 \qquad \text{and} \qquad N = N_1 + iN_2,$$

where

$$L_1 = \rho_0 \begin{bmatrix} 1 & -\dfrac{\omega_i}{2|\sigma|^2} r \dfrac{d\Omega}{dr} & -\dfrac{\omega_i}{2|\sigma|^2} \dfrac{dW}{dr} \\ -\dfrac{\omega_i}{2|\sigma|^2} r \dfrac{d\Omega}{dr} & 1 & 0 \\ -\dfrac{\omega_i}{2|\sigma|^2} \dfrac{dW}{dr} & 0 & 1 \end{bmatrix}$$

$$iL_2=\rho_0\begin{bmatrix} 0 & \dfrac{\omega_i}{2|\sigma|^2}r\dfrac{d\Omega}{dr} & \dfrac{\omega_i}{2|\sigma|^2}\dfrac{dW}{dr} \\ -\dfrac{\omega_i}{2|\sigma|^2}r\dfrac{d\Omega}{dr} & 0 & 0 \\ -\dfrac{\omega_i}{2|\sigma|^2}\dfrac{dW}{dr} & 0 & 0 \end{bmatrix}$$

$$M_1=\rho_0\begin{bmatrix} 0 & 2i\Omega & 0 \\ -2i\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$iM_2=\rho_0\begin{bmatrix} -2i\Omega\dfrac{\omega_i}{|\sigma|^2}r\dfrac{d\Omega}{dr} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N_1=\begin{bmatrix} \begin{matrix}-2\rho_0 r\Omega\dfrac{d\Omega}{dr}-\Omega^2 r\dfrac{d\rho_0}{dr}\\ -\dfrac{m^2B^2}{\mu r^2}+2\dfrac{B}{\mu r}\left(\dfrac{dB}{dr}-\dfrac{B}{r}\right)\end{matrix} & \begin{matrix}2i\dfrac{mB^2}{\mu r^2}\\ +\dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}r\dfrac{d\Omega}{dr}\end{matrix} & \dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}\dfrac{dW}{dr} \\ -2i\dfrac{mB^2}{\mu r^2}+\dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}r\dfrac{d\Omega}{dr} & -\dfrac{m^2B^2}{\mu r^2} & 0 \\ \dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}\dfrac{dW}{dr} & 0 & -\dfrac{m^2B^2}{\mu r^2} \end{bmatrix}$$

$$iN_2=\begin{bmatrix} -2i\dfrac{mB^2}{\mu r^2}\dfrac{\omega_i}{|\sigma|^2}\dfrac{d\Omega}{dr} & -\dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}r\dfrac{d\Omega}{dr} & -\dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}\dfrac{dW}{dr} \\ \dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}r\dfrac{d\Omega}{dr} & 0 & 0 \\ \dfrac{m^2B^2}{\mu r^2}\dfrac{\omega_i}{2|\sigma|^2}\dfrac{dW}{dr} & 0 & 0 \end{bmatrix}$$

Note that all the operators defined above (i.e., L_1, M_1, N_1, and L_2, M_2, N_2) are formally self-adjoint on E. (An operator X is said to be formally self-adjoint if $(\eta, X\xi)=(X\eta, \xi)$ for all η and ξ in the domain of X). The imaginary part of (2.16) then gives,

$$i\omega_i\left(G,\left[L_1+\frac{\sigma_r L_2}{\omega_i}+\frac{M_2}{\omega_i}-\frac{N_1}{|\sigma|^2}+\frac{\sigma_r}{\omega_i}\frac{N_2}{|\sigma|^2}\right]G\right)=0, \tag{2.17}$$

where σ_r is the real part of σ.

If we can show that the operator appearing in (2.17),

$$A = L_1 + \frac{\sigma_r L_2}{\omega_i} + \frac{M_2}{\omega_i} - \frac{N_1}{|\sigma|^2} + \frac{\sigma_r}{\omega_i} \frac{N_2}{|\sigma|^2}$$

is positive definite (or, negative definite), then ω_i must be zero, which is a contradiction of our initial assumption. Thus the sufficient condition of stability and the condition for positive definiteness of the operator A are equivalent. After a little more simplification, it follows that

$$A = \rho_0 \begin{bmatrix} 1+a+d-f & \bar{b}+g+db & \bar{c}+dc \\ b+\bar{g}+d\bar{b} & 1+d & 0 \\ c+d\bar{c} & 0 & 1+d \end{bmatrix} \tag{2.18}$$

where

$$a = \Omega^2 r \frac{1}{\rho_0} \frac{d\rho_0}{dr} \frac{1}{|\sigma|^2}, \tag{2.19}$$

$$b = \frac{i\sigma}{2|\sigma|^2} r \frac{d\Omega}{dr}, \tag{2.20}$$

$$c = \frac{i\sigma}{2|\sigma|^2} \frac{dW}{dr}, \tag{2.21}$$

$$d = \frac{m^2 B^2}{\mu\rho_0 r^2 |\sigma|^2}, \tag{2.22}$$

$$f = \frac{2}{\mu\rho_0|\sigma|^2} \frac{B}{r}\left(\frac{dB}{dr} - \frac{B}{r}\right) + \frac{2mB^2}{\mu\rho_0|\sigma|^2 r^2} \frac{\sigma_r}{|\sigma|^2} r \frac{d\Omega}{dr}, \tag{2.23}$$

$$g = -\frac{2\,\mathrm{im}\,B^2}{\mu\rho_0|\sigma|^2 r^2}. \tag{2.24}$$

The eigenvalues of (2.18) are the roots of

$$\det[\lambda I - A] = 0.$$

These eigenvalues are found to be

$$\lambda = 1 + d$$

and, the roots of

$$\lambda^2 - (a+2+2d-f)\,\lambda + (1+a+d-f)(1+d) - |\bar{b}+g+db|^2 - |\bar{c}+dc|^2 = 0.$$

The sufficient condition of stability then reduces to

$$a+2+2d-f>0 \tag{2.25a}$$

and

$$(1+a+d-f)(1+d)>|\bar{b}+g+db|^2+|\bar{c}+dc|^2. \tag{2.25b}$$

Using the inequality $|\bar{b}+g+db| \leqslant (1+d)|b|+|g|$, it can be easily shown that the inequality (2.25b) is satisfied if

$$(1+a+d-f)(1+d)>[(1+d)|b|+|g|]^2+(1+d)^2|c|^2. \tag{2.26}$$

By use of Eqs. (2.19)–(2.24), the conditions (2.25) and (2.26) are

$$\begin{aligned}&\Omega^2 r\frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{1}{|\sigma|^2}+\frac{2m^2B^2}{\mu\rho_0 r^2|\sigma|^2}-\frac{2}{\mu\rho_0|\sigma|^2}\frac{B}{r}\left(\frac{dB}{dr}-\frac{B}{r}\right)\\&\quad-\frac{2mB^2}{\mu\rho_0|\sigma|^2r^2}\frac{\sigma_r}{|\sigma|^2}r\frac{d\Omega}{dr}+2>0\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}&\left[1+\frac{m^2B^2}{\mu\rho_0 r^2|\sigma|^2}+\Omega^2 r\frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{1}{|\sigma|^2}-\frac{2}{\mu\rho_0|\sigma|^2}\frac{B}{r}\left(\frac{dB}{dr}-\frac{B}{r}\right)\right.\\&\quad\left.-\frac{2mB^2}{\mu\rho_0|\sigma|^2r^2}\frac{\sigma_r}{|\sigma|^2}r\frac{d\Omega}{dr}\right]\times\left[1+\frac{m^2B^2}{\mu\rho_0 r^2|\sigma|^2}\right]\\&\quad>\left[\left(1+\frac{m^2B^2}{\mu\rho_0 r^2|\sigma|^2}\right)\frac{1}{2|\sigma|}\left|r\frac{d\Omega}{dr}\right|+\frac{2|m|\,B^2}{\mu\rho_0|\sigma|^2r^2}\right]^2\\&\quad+\left(1+\frac{m^2B^2}{\mu\rho_0 r^2|\sigma|^2}\right)^2\left(\frac{1}{2|\sigma|}\left|\frac{dW}{dr}\right|\right)^2.\end{aligned} \tag{2.28}$$

The sufficient condition of stability (2.28) appears to be very complicated, especially due to the presence of m and σ. However, the case of rigid rotation yields some new interesting results. We now proceed to investigate this case in details.

By putting $d\Omega/dr=0$ and $W=0$, the condition (2.28) then reduces to

$$\begin{aligned}&\left[1+\frac{m^2B^2}{\mu\rho_0 r^2|\sigma|^2}+\Omega^2 r\frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{1}{|\sigma|^2}-\frac{2}{\mu\rho_0|\sigma|^2}\frac{B}{r}\left(\frac{dB}{dr}-\frac{B}{r}\right)\right]\\&\quad\times\left[1+\frac{m^2B^2}{\mu\rho_0 r^2|\sigma|^2}\right]>\frac{4m^2B^4}{\mu^2\rho_0^2|\sigma|^4r^4}.\end{aligned}$$

This will obviously be satisfied if

$$\Omega^2 r^3 \frac{d\rho_0}{dr} - \frac{2rB}{\mu}\left(\frac{dB}{dr} - \frac{B}{r}\right) \geqslant \frac{4B^2}{\mu} \tag{2.29}$$

throughout the flow region. Note that (2.27) is also satisfied by virtue of (2.29). Thus (2.29) is the sufficient condition for stability of flow with rigid rotation in presence of azimuthal field for arbitrary disturbances. This result does not also involve wavenumbers m and k. For homogeneous fluid, (2.29) reduces to

$$4B^2 + r^3 \frac{d}{dr}\left(\frac{B}{r}\right)^2 \leqslant 0. \tag{2.30}$$

It is pertinent here to mention that the sufficient condition for the same flow obtained by Ganguly and Gupta [11] did not include the case $|m| = 1$. The necessary condition of instability also follows from (2.30) as

$$r^3 \frac{d}{dr}\left(\frac{B}{r}\right)^2 + 4B^2 > 0$$

which shows that instability may occur even when $(d/dr)(B/r)^2 \leqslant 0$ for the mode $|m| = 1$. Acheson [8] obtained (page 536) for the same flow, considering slow hydromagnetic waves propagating in a rapidly rotating fluid, that $(d/dr)(B/r)^2$ has to be sufficiently large and positive to ensure instability. The consequence of such an analysis, however, led to the conclusion that the unstable modes are characterized by weak azimuthal propagation, which is perhaps untenable. It may be now interesting to see if condition (2.29) can be, in general, improved. To this end, we observe from (2.28) that, provided

$$\Omega^2 r \frac{d\rho_0}{dr} - \frac{r}{\mu}\frac{d}{dr}\left(\frac{B}{r}\right)^2 \geqslant 0$$

everywhere, the flow will be stable for all disturbances satisfying

$$\left(1 + \frac{m^2 B^2}{\mu\rho_0 r^2 |\sigma|^2}\right) > \frac{2|m|\, B^2}{\mu\rho_0 |\sigma|^2 r^2}. \tag{2.31}$$

One can easily verify that (2.31) is satisfied for $m = 0$ and $|m| \geqslant 2$. The mode $|m| = 1$ may, however, be unstable. A special case of this result for homogeneous flow is obtained by Ganguly and Gupta [11] and Acheson [8], the latter by employing local analysis. It appears that the general condition (2.29) cannot be improved further. From (2.28), we also recover that the flow is stable in absence of magnetic field if

$$\frac{\Omega^2 r}{\rho_0}\frac{d\rho_0}{dr} > \frac{1}{4}\left\{\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right\}. \tag{2.32}$$

This result, also obtained by Lalas [6] for compressible swirling flow, defines a Richardson condition for heterogeneous swirling flow against arbitrary disturbances. On the other hand, for the same problem, Kurzweg [7] obtained two conditions which are to be satisfied simultaneously to ensure stability. It is, however, interesting to note that Kurzweg's conditions are always satisfied if (2.32) holds.

In the case of rigid rotation in presence of azimuthal field, the growth rate of unstable modes can also be easily predicted by use of (2.28). In this case the necessary condition of instability is

$$\left[\frac{\Omega^2 r}{\rho_0}\frac{d\rho_0}{dr} - \frac{2}{\mu\rho_0}\frac{B}{r}\left(\frac{dB}{dr} - \frac{B}{r}\right)\right]\left(1 + \frac{m^2 B^2}{\mu\rho_0 r^2|\sigma|^2}\right) < \frac{4m^2B^4}{\mu^2\rho_0^2|\sigma|^2 r^4},$$

where the first two positive terms have been dropped. This inequality simplifies to

$$|\sigma| < |m|\frac{B}{\sqrt{\mu\rho_0}\, r}\left\{\left(\frac{4B^2}{\mu r^2} - \alpha\right)\Big/\alpha\right\}^{1/2},$$

where,

$$\alpha = \Omega^2 r\frac{d\rho_0}{dr} - \frac{2B}{\mu r}\left(\frac{dB}{dr} - \frac{B}{r}\right).$$

We assume $\alpha > 0$, i.e., the mode $|m| = 1$ can only be unstable. We also have $(4B^2/\mu r^2) - \alpha > 0$ for instability. Thus, we get the following bound for growth rate

$$\omega_i < \max\left[\frac{B}{\sqrt{\mu\rho_0}\, r}\left\{\left(\frac{4B^2}{\mu r^2} - \alpha\right)\Big/\alpha\right\}^{1/2}\right],$$

where maximum is over the domain $[a, b]$. This inequality places a limit on the growth rate when the flow is stable except for $|m| = 1$. Such a result is useful (Makov and Stepanyants [13]) for practical calculation and applications.

III. Swirling Flow in Presence of Axial Magnetic Field

Due to the presence of axial field, it is not possible to derive Richardson condition or such a sufficient condition of stability in this case. However, a similar analysis as in Section 2 yields bounds on growth rates and one can then qualitatively understand the effect of magnetic field. In this section, we give a short account of this.

The linearized perturbation equations are (Chandrasekhar [12])

$$\left\{\rho_0\sigma^2 - \rho_0 2\Omega r\frac{d\Omega}{dr} - \Omega^2 r\frac{d\rho_0}{dr} - \frac{k^2B_z^2}{\mu}\right\}\xi_r + 2i\rho_0\Omega\sigma\xi_\theta - \frac{dp}{dr} = 0, \tag{3.1}$$

$$-2i\rho_0\Omega\sigma\xi_r + \left(\rho_0\sigma^2 - \frac{k^2B_z^2}{\mu}\right)\xi_\theta + \frac{\mathrm{im}}{r}p = 0, \tag{3.2}$$

$$\left(\rho_0\sigma^2 - \frac{k^2B_z^2}{\mu}\right)\xi_z + ik\,p = 0. \tag{3.3}$$

Equations (2.4)–(2.10) are also applicable in the analysis of this section. Equation (2.11) changes to

$$(\sigma^2 L + \sigma M + 0)\,\sigma^{-1/2}G + F = 0, \tag{3.4}$$

where

$$0 = \begin{bmatrix} -2\rho_0 r\Omega\dfrac{d\Omega}{dr} - \Omega^2 r\dfrac{d\rho_0}{dr} - \dfrac{k^2B_z^2}{\mu} & 0 & 0 \\ \dfrac{k^2B_z^2}{\mu}\dfrac{\omega_i}{|\sigma|^2}r\dfrac{d\Omega}{dr} & -\dfrac{k^2B_z^2}{\mu} & 0 \\ \dfrac{k^2B_z^2}{\mu}\dfrac{\omega_i}{|\sigma|^2}\dfrac{dW}{dr} & 0 & -\dfrac{k^2B_z^2}{\mu} \end{bmatrix}$$

We now split 0 as $0_1 + i0_2$, where 0_1 and 0_2 are formally self-adjoint on E. 0_1 and 0_2 are found to be

$$0_1 = \begin{bmatrix} -2\rho_0 r\Omega\dfrac{d\Omega}{dr} - \Omega^2 r\dfrac{d\rho_0}{dr} - \dfrac{k^2B_z^2}{\mu} & \dfrac{1}{2}\dfrac{k^2B_z^2}{\mu}\dfrac{\omega_i}{|\sigma|^2}r\dfrac{d\Omega}{dr} & \dfrac{1}{2}\dfrac{k^2B_z^2}{\mu}\dfrac{\omega_i}{|\sigma|^2}\dfrac{dW}{dr} \\ \dfrac{1}{2}\dfrac{k^2B_z^2}{\mu}\dfrac{\omega_i}{|\sigma|^2}r\dfrac{d\Omega}{dr} & -\dfrac{k^2B_z^2}{\mu} & 0 \\ \dfrac{1}{2}\dfrac{k^2B_z^2}{\mu}\dfrac{\omega_i}{|\sigma|^2}\dfrac{dW}{dr} & 0 & -\dfrac{k^2B_z^2}{\mu} \end{bmatrix}$$

$$i0_2 = \begin{bmatrix} 0 & -\frac{1}{2}\frac{k^2B_z^2}{\mu}\frac{\omega_i}{|\sigma|^2}r\frac{d\Omega}{dr} & -\frac{1}{2}\frac{k^2B_z^2}{\mu}\frac{\omega_i}{|\sigma|^2}\frac{dW}{dr} \\ \frac{1}{2}\frac{k^2B_z^2}{\mu}\frac{\omega_i}{|\sigma|^2}r\frac{d\Omega}{dr} & 0 & 0 \\ \frac{1}{2}\frac{k^2B_z^2}{\mu}\frac{\omega_i}{|\sigma|^2}\frac{dW}{dr} & 0 & 0 \end{bmatrix}$$

It follows now easily that if (analysis is same as in Section 2)

$$B = L_1 + \frac{\sigma_r L_2}{\omega_i} + \frac{M_2}{\omega_i} - \frac{0_1}{|\sigma|^2} + \frac{\sigma_r}{\omega_i}\frac{0_2}{|\sigma|^2}$$

is positive definite (or, negative definite), then ω_i must be zero and hence the flow is stable. The expression for B is

$$B = \rho_0 \begin{bmatrix} 1+a+e & \bar{b}+eb & \bar{c}+ec \\ b+e\bar{b} & 1+e & 0 \\ c+e\bar{c} & 0 & 1+e \end{bmatrix}$$

where

$$e = \frac{k^2B_z^2}{\mu\rho_0|\sigma|^2}. \tag{3.5}$$

The eigenvalues of B are computed as

$$\lambda_1 = 1 + e \tag{3.6}$$

and the roots of

$$\lambda^2 - (a + 2e + 2)\lambda + (1+e)(1+e+a) - |b + e\bar{b}|^2 - |c + e\bar{c}|^2 = 0. \tag{3.7}$$

From (3.6) and (3.7), one can easily see that the sufficient condition of stability is

$$(1+e+a) > (|b|^2 + |c|^2)(1+e)$$

Substituting the values of a, b, c, e from Eqs. (2.19)–(2.21) and (3.5), this condition is

$$|\sigma|^4 + |\sigma|^2\left[\Omega^2 r\frac{1}{\rho_0}\frac{d\rho_0}{dr} + \frac{k^2B_z^2}{\mu\rho_0} - \frac{1}{4}\left\{\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right\}\right]$$
$$-\frac{1}{4}\left[\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right]\frac{k^2B_z^2}{\mu\rho_0} > 0. \tag{3.8}$$

A careful observation of (3.8) leads to some interesting concludions. Note that (3.8) is always violated by the mode $k \neq 0$ and $\sigma \to 0$. For $k = 0$ (azimuthal disturbances) we get back from (3.8) "Richardson" criterion for hydrodynamic stability. This is obvious from the physical reasoning that two-dimensional perturbations do not cause stretching of the magnetic lines of force and B_z does not at all appear in the perturbation equations in this case. Thus, the semicircle theorem for azimuthal disturbances (Lalas [6, p. 70]) also holds in presence of axial magnetic field. A necessary condition for instability follows from (3.8),

$$|\sigma|^2 < \left[\frac{1}{4}\left[\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right]\frac{k^2B_z^2}{\mu\rho_0}\right] \Big/ \left[\Omega^2 r\frac{1}{\rho_0}\frac{d\rho_0}{dr} - \frac{1}{4}\left\{\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right\}\right], \tag{3.9}$$

where the positive terms $|\sigma|^4$ and $|\sigma|^2k^2B_z^2/\mu\rho_0$ have been neglected. Equation (3.9) shows directly the dependence of growth rate on the wavelenght in the axial direction. The inclusion of the neglected terms changes the necessary condition of instability to

$$(|\sigma|^2 - \alpha)(|\sigma|^2 - \beta) \leqslant 0,$$

where

$$(\alpha, \beta) = \left[\gamma \pm \sqrt{\gamma^2 + \left\{\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right\}\frac{k^2B_z^2}{\mu\rho_0}}\right] \Big/ 2$$

with

$$\gamma = \frac{1}{4}\left\{\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right\} - \Omega^2 r\frac{1}{\rho_0}\frac{d\rho_0}{dr} - \frac{k^2B_z^2}{\mu\rho_0}.$$

Since $|\sigma|^2 - \beta \geqslant 0$, the necessary condition of instability reduces to

$$|\sigma|^2 - \alpha \leqslant 0.$$

Thus, we get the following bound on growth rate

$$\omega_i^2 \leqslant \left[\gamma + \left\{\gamma^2 + \left[\left(r\frac{d\Omega}{dr}\right)^2 + \left(\frac{dW}{dr}\right)^2\right]\frac{k^2B_z^2}{\mu\rho_0}\right\}^{1/2}\right] \Big/ 2. \tag{3.10}$$

One can easily see from (3.10) that the disturbances having very long wavelengths in the axial direction have nearly the same limit on growth rate as in the hydrodynamic case. The effect of the magnetic field on growth rate is pronounced for disturbances having short wavelengths.

Some more general results on growth rate follow by writing (3.8) in the form

$$\frac{\Omega^2 r}{|\sigma|^2}\frac{1}{\rho_0}\frac{d\rho_0}{dr}+\frac{k^2B_z^2}{\mu\rho_0|\sigma|^2}+1$$
$$>\frac{1}{4|\sigma|^2}\left[\left(r\frac{d\Omega}{dr}\right)^2+\left(\frac{dW}{dr}\right)^2\right]\times\left[1+\frac{k^2B_z^2}{\mu\rho_0|\sigma|^2}\right]. \tag{3.11}$$

The necessary condition of instability follows then (the first term not being included)

$$|\sigma|^2<\frac{1}{4}\left[\left(r\frac{d\Omega}{dr}\right)^2+\left(\frac{dW}{dr}\right)^2\right]$$

from which, we get the following bound for the growth rate

$$\omega_i^2<\max\frac{1}{4}\left[\left(r\frac{d\Omega}{dr}\right)^2+\left(\frac{dW}{dr}\right)^2\right], \tag{3.12}$$

where the maximum is taken over the domain $[a, b]$. The condition (3.12) also holds for homogeneous fluid ($\rho_0=$ const.) and does not involve the wavenumbers explicitly.

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